International Journal of Applied Mathematics
and Statistical Sciences (IJAMSS)
ISSN (P): 2319–3972; ISSN (E): 2319–3980
Vol. 10, Issue 2, Jul–Dec 2021; 15–24

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

REVIEW PAPER ON THE RUNGE-KUTTA METHODS TO STUDY NUMERICAL SOLUTIONS OF INITIAL VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

Pushap Lata Sharma & Ashok Kumar

Research Scholar, Department of Mathematics, Himachal Pradesh University, Summer Hill, Shimla, India

ABSTRACT

The main purpose of this article is to review the work on Runge-Kutta Methods to study of Numerical Solutions of Initial Value Problems in Ordinary Differential Equations during the period 1983 to the present 2020. A very comprehensive and detailed review on Runge-Kutta Methods has been compiled by Hulls from 1983 to 1996 and the authors of present article have reproduced it as such with gratitude and sincere acknowledgment. The status on this subject from 1983 to 2020 has been compiled by the present authors for the convenience of the new researchers entering in this field

KEYWORDS: *Initial Value Problem, Numerical Solutions, Ordinary Differential Equations, Runge-Kutta Methods*

Article History

Received: 04 Jul 2021 | Revised: 15 Sep 2021 | Accepted: 20 Sep 2021

INTRODUCTION

Differential Equations help us to understand phenomena that involves rate of change. These equations are the most important mathematical tools in generating models in Engineering, Biological Sciences and Physical Sciences. The differential equations are applied as mathematical representations of many actual world problems. For example, differential equations help us understand disease spread, weather and climate prediction, traffic flow, financial markets, population growth, water pollution, chemical reactions, suspension bridges, brain function, tumor growth, radioactive decay, electrical circuits, planetary motion and vibration of guitar strings. An important type of problem that we must solve when we study Ordinary Differential Equations is an initial value problem. An "initial value problem" is an ordinary differential equation together with the conditions imposed on the unknown function and the values of its derivatives in a single number is called an Initial Value Problem. At some point, the initial value problem is too complicated to solve exactly, and one of two approaches is taken to approximate the solution. The first approach is to simplify the differential equation to one that can be solved exactly and then use the solution of the simplified equation to approximate the solution to the original equation. Numerical Methods are generally used to solve mathematical problems that are formulated in science and engineering, where it is difficult to get an exact solution, only a few differential equations can be solved analytically. There are several analytical methods for solving ordinary differential equations. But there are a lot of ordinary differential equations that cannot be solved analytically. At that stage we have to use the Numerical Methods to solve this type of Ordinary Differential Equations. There are several Numerical Methods for solving Initial Value Problems. While studying literature, we came across the numerous works of numerical solution of Initial Value Problems applying the Runge-Kutta first-order method, the Runge-Kutta second and third order methods, and the Runge-Kutta fourth-order method have been carried out.

Numerous authors have tried to solve initial value problems to obtain good precision using the methods mentioned above. Hull et.al [1996] worked on in detailed about the literature of the Runge-kutta method from 1957 to 1996. Further review of literature from 1997 to 2020 has been compiled by the present authors.

Numerical Method

Numerical method forms an important part of solving initial value problems in ordinary differential equations, most especially in cases where there is no closed form solution. Next we will present numerical Runge-Kutta Methods.

Runge-Kutta Method

Runge-Kutta Method is a technique for approximating the solution of ordinary differential equation. This technique was developed around 1900 by the mathematicians Carl Runge and Wilhelm Kutta. Runge-Kutta Method is popular because it is efficient and used in most computer programs for differential equation.

The following are the orders of Runge-Kutta Method as listed below:

- Runge-Kutta Method of order one is called Euler's Method.
- Runge-Kutta Method of order two is the same as modified Euler's or Heun's Method.
- Runge-Kutta Method of order three.
- The fourth order Runge-Kutta Method called classical Runge-Kutta Method.

Solution of Ordinary Differential Equations

Runge'sKutta Method: By this method we are able to find the increment k of y corresponding to an increment h of x from$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$

Runge-Kutta Formulae

IstOrder :

- Euler's Method is Runge-Kutta Method of 1st Order
- $y_{n+1} = y_n + hf(x_n, y_n)$

IIndOrder :

- $k_1 = hf(x_n, y_n)$
- $k_2 = h[x_n + h, y_n + hf(x_n, y_n)]$
- $y_{n+1} = y_n + \frac{1}{2}[k_1 + k_2]$

IIIrdOrder :

- $k_1 = hf(x_n, y_n)$
- $k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$
- $k_3 = hf(x_n + h, y_n + 2k_2 - k_1)$

- $y_{n+1} = y_n + \frac{h}{6}[k_1 + 4k_2 + k_3]$

IVth Order

- $k_1 = hf(x_n, y_n)$
- $k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$
- $k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$
- $k_4 = hf(x_n + h, y_n + k_3)$
- $y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$, where, n= $0,1,2,3, \ldots \ldots N-1$ where N is the number of subintervals.

REVIEW OF LITERATURE FROM 1983 TO 2020

A new theme, which has led to several new initiatives during this period, is the development of continuous methods. A major new implementation project is also being undertaken during this period, along with expanded testing facilities for assessment and comparison of new methods. Another theme is the development of parallel methods. Much of the work in this period is surveyed in a 1994 paper by Enright, Higham, Owren and Sharp [1994].

Owren and Zennaro [1991,1992], Muir and Owren [1993], and Verner [1993]. Effectiveness theory has been extended to this class of methods and continuous Runge-Kutta formulas of orders four to eight have been analyzed from this point of view by Enright [1993] Continuous extensions of Runge-Kutta formulas have also been applied to singular initial-value problems by Enright and Suhartanto [1992].An implementation of an explicit Runge-Kutta code for initial-value ODE problems, known as TERK [1986], based on these new results has been under development for some time, and is now nearing completion. The work on TERK also motivated research on global error estimates by Peterson [1986] and Higham [1991] and on stiffness detection by Robertson [1986]; it is planned to incorporate these capabilities in a future version of TERK. Some special Runge-Kutta formulas have been developed and implemented for boundary-value problems by Muir [1984] and Enright and Muir [1986]. Still others have been developed and implemented for first-order problems by Sharp [1989], and second-order problems by Sharp [1987] and Sharp and Fine [1987,1992] (but these last four do not involve continuous extensions). Investigations related to the assessment and comparison of numerical methods have continued. Sharp [1991] has developed new formats for viewing and reporting the results of comparisons and he has introduced an option into DETEST to report statistics in this format. For the comparison of different continuous methods, an extended version of DETEST, called X-DETEST, has been written which reports other measures of error such as the maximum defect and the maximum global error of the continuous approximation. DDETEST, another test program that measures performance based on maximum global error and the size of the defect, is being developed for the assessment of methods for delay differential equations. It uses a collection of twenty-one test problems, and it stores the "true solution" as piecewise polynomials, obtained using high-precision arithmetic. Several aspects of parallelism have been considered during this period. The potential for parallelism in standard Runge-Kutta Methods has been studied by Jackson and Norsett [1995]: both negative and positive results are presented. Many of the negative results are based on a theorem that bounds the order of a Runge-Kutta formula in terms of the minimal polynomial associated with its coefficient matrix. The positive results are largely examples of prototypical formulas which offer a potential for effective "coarse-grain" parallelism on machines with a few processors.Runge-Kutta predictor--corrector methods have been discussed by Enenkel [1988], and analysed by Jackson, Kvmmo and Norsett [1994]. The results in the last two papers are of value not only for the development of parallel ODE codes, but also for the efficient, reliable and robust implementation of implicit Runge-Kutta methods on standard sequential machines.Onumanyiet.al.[1999] studied continuous finite difference approximations for solving differential equations". Gander [1999] gave a thoughtful consideration on the derivation of numerical methods using computer algebra.Hong[2000] worked on "The calculation of global error for initial value problem of ordinary differential equations". Ling [2000] studied some Application of Runge-Kutta–Merson Algorithm for Creep Damage Analysis. Further, Boyce [2000] worked on Elementary Differential Equations and Boundary Value Problems. Runge–Kutta with higher order derivative approximations was appeared in a paper by Goeken and Johnson [2000].Pimenov [2001] provides general linear methods for the numerical solution of functional – Differential Equations. Bernardo and Wang [2001] gave some aspects about Runge-Kutta discontinuous Galerkin Methods for convection-dominated problems. Awoyemi [2001] discovered a new sixth-order algorithm for general second order Ordinary Differential Equation. Fredebeul et.al. [2002] worked on multiple order double output Runge–Kutta Fehlberg formulae and some strategies for its efficient applications. Murugesan[2002] have studied a fourth order embedded Runge–Kutta Method based on arithmetic and centroidal means with error control. Awoyemi [2003], studied A P-stable linear multistep method for solving general third order ordinary differential equations. Also, Butcher [2003] give detailed studies on Numerical Methods for Ordinary Differential Equations. Biazar et.al.[2004]obtained solution of the system of ordinary differential equations by Adomian decomposition method. Ruuth [2004] studied the global optimization of explicit strong–stability preserving Runge – Kutta methods. Awoyemi[2005]studied a class of hybrid collocation methods for third-order ordinary differential equation. Further, Mathews [2005] gave many results about the book Numerical Methods for Mathematics. Majid[2006]worked on higher order systems of Ordinary Differential Equations block backward differentiation formula for solving first-order Ibrahim [2007] worked on "Implicitr-point block backward differentiation formula for solving first-order stiff ODEs," Bosede et. al. [2012] gave some numerical methods in their famous paper entitled "On Some Numerical Methods for Solving Initial Value Problems in Ordinary Differential Equations". Also, Ogunrinde [2012] studied some numerical methods for solving Initial Value Problems in Ordinary Differential equations. Rabiei [2012] improved the Fifth-order Runge-Kutta method for solving ordinary differential equation. Islam [2015] analysed the accuracy of Numerical Solutions of Initial Value Problems (IVP) for Ordinary Differential Equations (ODE). He also gave the accurate solutions of Initial Value Problems for Ordinary Differential Equations with the fourth order Runge Kutta Method. Further, Islam in [2015], performed a comparative study on numerical solutions of initial value problems (IVP) for ordinary differential equations (ODE) with Euler and Runge-Kutta methods. Gowri et.al [2017] discussed about Runge-Kutta Fourth Order Method with Differential Equations and its Application. The Differential Equation problems hasbeen solved by Runge-Kutta fourth order method and application problem are discussed with Runge-Kutta fourth order. He concluded that Runge- Kutta fourth order Method gave more accurate results. Sadiq et.al [2017] worked on using fourth order Ruge-Kutta Method to solve Lü Chaotic System. He observed that the accuracy of Runge-Kutta fourth order method solution can be increased by lessening the time step and shows that Runge-Kutta fourth order method successfully to solve the Lu system. It has been determined the accuracy of method using symmetrical times. Hossain[2017] worked on a paper entitled "A study on the Numerical Solutions of Second Order Initial Value Problems (IVP) for Ordinary Differential Equations with Fourth Order and Butcher's Fifth Order Runge-Kutta Methods". Hamed, [2017] gave the accuracy of Euler and Modified Euler technique for first order ordinary differential equations with initial conditions.Samsudinet.al.[2018] worked on "Cube Arithmetic: Improving Euler Method for Ordinary Differential Equation Analysis and Comparative Study of Numerical Solutions of Initial Value Problems (IVP) in Ordinary Differential Equations (ODE) With Euler and Runge-Kutta Methods. Anthony [2019] et.al. Analysed and Compared the Numerical Solutions of Initial Value Problems. He compared the performance and the computational effort of the two methods. In addition, to achieve more accuracy in the solutions, the step size needs to be very smallN Jamali [2019] produced the analysis and the comparative study of Numerical Methods to solve Ordinary Differential Equations with Initial Value Problems. Mohammad et.al [2020] worked on a Comparative exploration on different Numerical Methods for solving Ordinary Differential Equations. In this paper, the Initial Value Problem of Ordinary Differential Equations has been solved by using different Numerical Methods namely Euler's method, Modified Euler method, and Runge-Kutta Methods. He analysed to determine the accuracy level of each Method. By using MATLAB Programming language first we find out the approximate numerical solution of some ordinary differential equations and then to determine the accuracy level of the proposed methods we compare all these solutions with the exact solution. It is observed that numerical solutions are in good agreement with the exact solutions and numerical solutions become more accurate when taken step sizes are very much small. Mohammad et.al [2020] worked on a comparative exploration on different numerical method for solving ordinary differential equations. In this paper, the initial value problem of Ordinary Differential Equations has been solved by using different Numerical Methods namely Euler's method, Modified Euler method, and Runge-Kutta method. Here all of the three proposed methods have to be analysed to determine the accuracy level of each method. By using MATLAB Programming language first we find out the approximate numerical solution of some ordinary differential equations and then to determine the accuracy level of the proposed methods we compare all these solutions with the exact solution. It is observed that numerical solutions are in good agreement with the exact solutions and numerical solutions become more accurate when taken step sizes are very much small. Lastly, the error of each proposed method is determined and represents them graphically which reveals the superiority among all the three methods. We fund that, among the proposed methods Runge-Kutta 4th order method gives the accurate result and minimum amount of error.

CONCLUSIONS

We have tried to go through each and every research article on this subject till date. It is very clear from this compilation majority of the work on this topic comes from the workers overseas they have dealt mainly with the accuracy and comparative analysis on Runge-Kutta first order second order third order and fourth order method. This review reflects that there is hardly any report of work available in the Indian scenario. Hence there is strong need of taking up detailed studies on this problems which will prove beneficial in solving the initial value problems.

REFERENCES

1. *Akanbi, M.A. (2010). Propagation of Errors in Euler Method, Scholars Research Library.Archives of Applied Science Research, 2, 457-469.*
2. *Atkinson, K., Han, W. and Stewart, D. (2009).Numerical Solution of Ordinary Differential Equations. New Jersey: John Wiley & Sons, Hoboken: 70-87.*
3. *Awoyemi, D. (2001). A New Sixth-Order Algorithm for General Second Order Ordinary Differential Equation. International Journal of Computer Mathematics, 77(1), 117-124.*
4. *Awoyemi, D. (2003). A P-stable Linear Multistep Method for Solving General Third Order Ordinary Differential Equations.. International Journal of Computer Mathematics, 80, 985-991.*
5. *Bosede, O., Emmanuel, F. and Temitayo, O. (2012). On Some Numerical Methods for Solving Initial Value Problems in Ordinary Differential Equations. IOSR Journal of Mathematics (IOSRJM) Vol. 1 (3), 25-31.*

6. *Boyce W, DiPrima R, (2000). Elementary Differential Equations and Boundary Value Problems. New York: John Wiley & Sons, Inc.: 419-471.*
7. *Broderick, N.G.R. de Sterke, C. M. and Jackson, K. R. (1994). Coupled mode equations with free carrier effects: a numerical solution, J. Optical Quantum Electronics 26, 219-234.*
8. *Burden, R.L. and Faires, J.D. (2002). Numerical Analysis. Bangalore, India.*
9. *Butcher, J. (2003). Numerical Methods for Ordinary Differential Equations. West Sussex: John Wiley & Sons Ltd. 45-95.*
10. *Butcher, John C. (2008). Numerical Methods for Ordinary Differential Equations, New York: John Wiley & Sons, ISBN 978-0-470-72335-7.*
11. *Chan, T.F., Jackson, K.R. & Zhu, B. (1983). Alternating-direction incomplete factorizations, SlAM J. Numer.Anal. 20, 239-257.*
12. *Chan, T. F. and Jackson, K. R. (1984). Nonlinearly preconditioned Krylov subspace methods for discrete-Newton algorithms, SlAM J. Sci. Statist.Comput. 5, 533-542.*
13. *Chan, T. F. and Jackson, K. R. (1986). The use of iterative linear-equation solvers in codes for large systems of stiff IVPs for ODEs, SIAM J. Sci. Statist.Comput. 7, 378-417.*
14. *Chapra, S. and Canale, R. (2006). Applied Numerical Methods with MATLAB for Engineers and Scientists, 6 Ed. Boston: McGraw Hill: 707-742.*
15. *Cockburn, B. and Wang, C. S. (2001). Runge-Kutta discontinuous Galerkin Methods for convection-dominated problems. Journal of Scientific Computing, 16, 173-261*
16. *Enenkel, R. F. (1988). Implementation of parallel predictor-corrector Runge-Kutta methods, Master's thesis, University of Toronto.*
17. *Enenkel, R. F. and Jackson, K. R. (1995). DIMSEMs-diagonally implicit single-eigenvalue methods for the numerical solution of stiff ODEs on parallel computers: A preliminary report.*
18. *Enenkel, R.F. (1996). DIMSEMs---diagonally implicit single-eigenvalue methods for the numerical solution of stiff ODEs on parallel computers, Ph.D. thesis, University of Toronto.*
19. *Enright, W. H. (1989). Analysis of error control strategies for continuous Runge-Kutta methods, SlAM J. Numer. Anal. 26, 588-599.*
20. *Enright, W. H. (1989). A new error-control for initial value solvers, Appl. Math.Comput. 31, 288-301.*
21. *Enright, W. H. and Higham, D. J. (1991). Parallel defect control, BIT 31, 647-663.*
22. *Enright, W. H. (1993). The relative efficiency of alternative defect control schemes for high order Runge-Kutta formulas, SlAM J. Numer. Anal. 30, 1419-1445.*
23. *Enright, W. H. and Hayashi, H. (1996). Convergence analysis for the numerical solution of retarded and neutral delay differential equations by continuous numerical methods.*

24. *Enright, W. H. and Hayashi, H. A delay differential equation solver based on a continuous Runge-Kutta method, SIAM J. Sci. Comput., submitted.*
25. *Enright, W.H., Jackson, K.R., Norsett, S.P. and Thomsen, EG. (1986). Interpolants for Runge-Kutta formulas, ACM Trans. Math. Software 12, 193-218.*
26. *Enright, W.H. and MacDonald, C. (1992). The practical implications of order reduction for implicit Runge-Kutta methods, Numer. Algorithms 2, 351-370.*
27. *Enright, W.H. and Muir, E. (1986). Efficient classes of Runge-Kutta methods for two point boundary value problems, Computing 37, 315-334.*
28. *Enright, W. H. Higham, D. J. Owren, B. and Sharp, P.W. (1994).A survey of the explicit Runge-Kutta method, Technical Report No. 291, Department of Computer Science, University of Toronto.*
29. *Enright, W.H. and Hu, M. (1995). Interpolating Runge-Kutta methods for vanishing delay differential equations, Computing 55, 223-236.*
30. *Enright, W.H., Jackson, K.R., Norsett, S.P. and Thomsen, EG. (1988). Effective solution of discontinuous IVPs using a Runge-Kutta formula pair with interpolants, Appl. Math.Comput. 27, 313-335.*
31. *Enright, W.H. and Muir, E. (1996). A Runge-Kutta type boundary value ODE solver with defect control, SIAM J. Sci. Comput. 17, 479-497.*
32. *Enright, W.H. and Pryce, J.D. (1983). Two Fortran packages for assessing initial value methods, Technical Report No. 167, Department of Computer Science, University of Toronto.*
33. *Enright, W.H. and Pryce, J.D. (1987). Two FORTRAN packages for assessing initial value methods, ACM Trans. Math. Software 13, 1-27.*
34. *Enright, W.H. and Suhartanto, H. (1992). Detecting and locating a singular point in the numerical solution of 1VPs for ODEs, Computing 48, 161-175.*
35. *Fairgrieve,T.E., Enright, W.H. and Jackson, K.R. (1996). A user's guide to TERK (1996).*
36. *Fine, J.M. (1987). Interpolants for Runge-Kutta-Nystr6m methods, Computing 39 27-42.*
37. *Fredebeul, C. Kornmair, D. and Muller, M. W. (2002). Multiple Order Double Output Runge-KuttaFehlberg Formulae: Strategies for Efficient Application. Journal of Computational and Applied Mathematics,*
38. *Gerald, C.F. and Wheatley, P.O. (2002). Applied Numerical Analysis. Pearson Education, India.*
39. *Goeken, D., & Johnson, O. (2000). Runge–Kutta with higher order derivative approximations.Applied numerical mathematics, 34(2-3), 207-218.*
40. *Gowri, P., Priyadharsini, S. and Maheswari, T. (2017). A Case Study on Runge-Kutta Fourth Order differential Equations and Its Application. Imperial Journal of Interdisciplinary Research, 3.*
41. *Hamed, A. B., Alrhaman, I. Y. and Sani, I. (2017). The accuracy of Euler and modified Euler technique for First Order Ordinary Differential Equations with initial conditions. American journal of Engineering Research (AJER), Vol. 6(9): 334-338.*

42. *Hayashi, H. (1996). Numerical solution of retarded and neutral delay differential equations using continuous Runge- Kutta methods, Ph.D. thesis, University of Toronto.*

43. *Higham, D.J. (1991). Global error versus tolerance for explicit Runge-Kutta methods, IMA J. Numer.Anal. 11, 457-480.*

44. *Higham, D.J. (1991). Runge-Kutta defect control using Hermite-Birkoff interpolation, SIAM J. Sci. Statist.Comput. 12, 991-999.*

45. *Hong-Yi, L. (2000). The calculation of global error for initial value problem of ordinary differential equations.International journal of computer mathematics, 74(2), 237-245.*

46. *Hossain, Md. S., Bhattacharjee, P.K. and Hossain, Md. E. (2013). Numerical Analysis.Titas Publications, Dhaka.*

47. *Hossain, M. J., Alam, M. S. &Hossain, M. B. (2017). A study on the Numerical Solutions of Second Order Initial Value Problems (IVP) for Ordinary Differential Equations with Fourth Order and Butcher's Fifth Order Runge-KuttaMthods. American Journal of Computational and Applied Mathematics, 7(5), 129-137.*

48. *Hossen, M., Ahmed, Z., Kabir, R. and Hossain, Z. (2019). A Comparative Investigation on Numerical Solution of Initial Value Problem by Using Modified Euler Method and Runge-Kutta Method. IOSR Journal of Mathematics, 15, 40-45.*

49. *Hull, T.E.,Enright, W.H., Jackson, K.R. (1996). Runge-Kutta research at Toronto.Applied Numerical Mathematics 22,225-236.*

50. *Ibrahim, Z.B., Othman, K. I. &Suleiman, M. (2007). Variable Step Block Backward Differentiation Formula for Solving First Order Stiff ODEs. Proceedings of the World Congress on Engineering, 2.*

51. *Islam, M. A. (2015). Accuracy Analysis of Numerical solutions of initial value problems (IVP) for ordinary differential equations (ODE). IOSR Journal of Mathematics, 11(3), 18-23.*

52. *Islam, M. A. (2015). Accurate solutions of initial value problems for ordinary differential equations with the fourth order RungeKutta method. Journal of Mathematics Research, 7(3), 41.*

53. *Islam, M. A. (2015). A Comparative Study on Numerical Solutions of Initial Value Problems (IVP) for Ordinary Differential Equations (ODE) with Euler and RungeKutta Methods.American Journal of Computational Mathematics, 5(03), 393.*

54. *Jackson, K.R. (1991). A survey of parallel numerical methods for initial value problems for ordinary differential equations, IEEE Trans. Magnetics 27, 3792-3797.*

55. *Jackson, K.R. (1994). An application of Runge-Kutta predictor-corrector methods to two system of hyperbolic equations arising in optics, in: W.E Ames, ed., Fourteenth IMACS World Congress, Atlanta, GA, 1329-1331.*

56. *Jackson, K.R. (1996). The numerical solution of large systems of stiff IVPs for ODEs, Appl. Numer.Math. 20, 1-16.*

57. *Jackson, K.R., Kvarno, A. and Norsett, S.P. (1994). The use of Butcher series in the analysis of Newton-like iterations in Runge-Kutta formulas, Appl. Numer.Math. 15, 341-356.*

58. *Jackson, K.R. and Norsett, S.P. (1995). The potential for parallelism in Runge-Kutta methods, Part 1: RK formulas in standard form, SIAM J. Numer. Anal. 32, 49-82.*
59. *Jackson, K.R. and Pancer, R.N. (1992). The parallel solution of ABD systems arising in numerical methods for BVPs for ODEs, Technical Report No. 255, Department of Computer Science, University of Toronto.*
60. *Jackson, K.R. and Seward, W.L. (1993). Adaptive linear equation solvers in codes for large stiff systems of ODEs, SIAM J. Sci. Comput. 14, 800-823.*
61. *Jamali, N. (2019). Analysis and Comparative Study of Numerical Methods to Solve Ordinary Differential Equation with Initial Value Problem.International Journal of Advanced Research (IJAR). Vol. 7(5): 117-128.*
62. *Kendall, E. Atkinson, (1998). An Introduction to Numerical Analysis.John Wiley & Sons.*
63. *Ling, Xiang, Tu, Shan – Tung, Gong and Jain – Ming (2000). Application of Runge–Kutta – Merson Algorithm for Creep Damage Analysis.International Journal of Pressure Vessels and Piping, Vol. 77, No. 5, 243 -248, Compendex.*
64. *Arefin, A. M, Gain, B., Karim, R., Hossain, S.[2020]. A Comparative exploration on different Numerical Methods.Journal of Mechanics of continua and Mathematical Sciences.Vol.15(12):1-11*
65. *Mathews, J.H. (2005). Numerical Methods for Mathematics, Science and Engineering. Prentice-Hall, India.*
66. *Majid,Z.A., Suleman, M.B. and Omar, Z. (2006). 3-point Implicit Block Method for Solving Ordinary Differential Equations. The Bulletin of the Malaysian Mathematical Society Series, 21(1), 1-9.*
67. *Muir, E. and Owren, B. (1993). Order barriers and characterizations of continuous mono-implicit Runge-Kutta schemes, Math. Comp. 61, 675-699.*
68. *Muir, EH. (1984). Implicit Runge-Kutta methods for two-point boundary value problems, Ph.D. thesis, University of Toronto.*
69. *Murugesan, K. (2002). Fourth Order Embedded Runge–Kutta Method Based on Arithmetic and Centroidal Means with Error Control. International Journal of Computer Mathematics, 79(2), 247-269.*
70. *Nguyen, H. (1995). Interpolation and error control schemes for algebraic differential equations using continuous implicit Runge-Kutta methods, Ph.D. thesis, University of Toronto.*
71. *Ogunrinde, R. B., Fadugba, S. E., &Okunlola, J. T. (2012). On some numerical methods for solving initial value problems in ordinary differential equations. IOSR Journal of Mathematics, 1(3), 25-31.*
72. *Okeke, A.A., Tumba Pius, Anorue, O.F. and Dauda A.A. (2019). Analysis and Comparative Study of Numerical Solutions of Initial Value Problems in ODEs WithEuler and Runge-Kutta Methods. American Journal of Engineering Research, 8, 40-53.*
73. *Owren, B. and Zennaro, M. (1991). Order barriers for continuous explicit Runge-Kutta methods, Math. Comp. 56, 645-661.*
74. *Owren, B. and Zennaro, M. (1992). Derivation of efficient continuous explicit Runge-Kutta methods, SIAM J. Sci. Statist.Comput. 13, 1488-1501.*

75. *Peterson, P.J. (1986). Global error estimates using defect correction techniques for explicit Runge-Kutta methods, Master's thesis, University of Toronto.*

76. *Pimenov, V. G. (2001). General Linear Methods for the Numerical Solution of Functional – Differential Equations. Differential Equations, Vol. 37, No. 1.*

77. *Rabiei, F., & Ismail, F. (2012). Fifth-order Improved Runge-Kutta method for solving ordinary differential equation. Australian Journal of Basic and Applied Sciences, 6(3), 97-105.*

78. *Robertson, B.C. (1986). Detecting stiffness with explicit Runge-Kutta formulas, Master's thesis, University of Toronto.*

79. *Samsudin, N., Yusop, N. M. M., Fahmy, S., &bintiMokhtar, A. S. N. (2018). Cube Arithmetic: Improving Euler Method for Ordinary Differential Equation Using Cube Mean. Indonesian Journal of Electrical Engineering and Computer Science, 11(3), 1109-1113.*

80. *Sastry, S.S. (2000). Introductory Methods of Numerical Analysis. Prentice-Hall, India.*

81. *Sharp, P.W. (1987). ERNY-an explicit Runge-Kutta-Nystrom integrator for second order initial value problems, Technical Report No. 199, Department of Computer Science, University of Toronto.*

82. *Sharp, P.W. (1989). New low order explicit Runge-Kutta pairs, Technical Report No. 222, Department of Computer Science, University of Toronto.*

83. *Sharp, P.W. (1991). Numerical comparison of explicit Runge-Kutta pairs of orders four through eight, ACM Trans. Math. Software 17, 387-409.*

84. *Sharp, P.W. and Fine, J.M. (1987). User guide for ERNY--an explicit Runge-Kutta-Nystrom integrator for second order initial value problems, Technical Report No. 198, Department of Computer Science, University of Toronto.*

85. *Sharp, P.W. and Fine, J.M. (1992). Some Nystrom pairs for the general second order initial value problem, J. Comput. Appl. Math. 42, 279-291.*

86. *Suhartanto, H. (1990). A new approach for detecting singular points in the numerical solution of initial value problems, Master's thesis, University of Toronto.*

87. *Verner, J.H. (1993). Differentiable interpolants for higher-order Runge-Kutta methods, SIAM J. Numer.Anal. 39, 1446-1466.*